## § 14 Local L-factors.

$F$ local non-arch. $\quad G = GL_2(F)$

$\pi$ irred adm. (exclude character case) $\rightsquigarrow$ has Whittake model $\mathcal{W}(\pi)$, Kirillov model $\mathcal{K}(\pi)$.

Form local zeta integrals $\quad \chi$ char of $GL_1(F)$, $W \in \mathcal{W}(\pi)$

$$Z(s, W, \chi) = \int_{F^\times} W\begin{pmatrix} x & \\ & 1 \end{pmatrix} \chi(x) |x|^{s-\frac{1}{2}} d^\times x$$

① Abs conv. for $\operatorname{Re} s \gg 0$

② Merom. continuation

③ Functional equation

<u>Local L-factor</u> $L(s, \pi\times\chi) :=$ g.c.d. of $\underbrace{\{ Z(s, W, \chi) \mid W \in \mathcal{W}(\pi) \}}_{\text{a non-zero frac. ideal of } \mathbb{C}[q^s, q^{-s}]}$.

also written as $L(s, \pi, \chi)$

$L(s, \pi\times\chi)$ is required to be of the form $P(q^{-s})^{-1}$ w/ $P$ a polynomial, $P(0)=1$. so $L(s, \pi\times\chi)$ uniquely determined.

Some rmks on generalisation

$GL_1$ - case (Tate)

$$\chi \rightsquigarrow \left\{ \overset{Z^{\mathrm{Tate}}(s,\chi,\varphi)}{\overset{\parallel}{\int_{F^\times} \varphi(x)\,\chi(x)\,|x|^s\,d^\times x}} \quad \varphi : \text{Schwartz fun. on } F \right\} \subset \mathbb{C}(q^s, q^{-s}) \overset{\text{g.c.d}}{\rightsquigarrow} L(s,\chi).$$

$GL_n$ - case (Godement - Jacquet) Also for $GL_n(D)$ ← division alg./$F$.

$$\pi \rightsquigarrow \left\{ \begin{array}{l} \text{zeta integral formed from Schwartz fun. on } \mathrm{Mat}_n \\ \text{matrix coeff of } \pi, \ |\det|^s \end{array} \right\} \xrightarrow{\text{g.c.d}} L(s,\pi)$$

Current case is actually $GL_2 \times GL_1$ - case.

$GL_n \times GL_m$ - case (Jacquet-Langlands, JPSS, Cogdell-PS)
(Ref: Cogdell notes on L-functions for $GL(n)$)

$$\pi_1 \times \pi_2 \rightsquigarrow \left\{ \begin{array}{l} \text{zeta integrals formed from } W(\pi_1), W(\pi_2) \\ |\det|^s \end{array} \right\} \xrightarrow{\text{g.c.d}} L(s, \pi_1 \times \pi_2)$$

Computation (use Kirillov model $K(\pi)$ )

(A) $\pi$ supercuspidal.

Want g.c.d of $Z(s,\xi,\chi) := \int_{F^\times} \xi(x)\,\chi(x)\,|x|^{s-\frac{1}{2}}\,d^\times x$. (*) $\forall\, \xi \in K(\pi)$

Recall space of $K(\pi)$ is $\mathcal{S}(F^\times)$.

(*) always abs. conv. $Z(s,\xi,\chi)$ entire in $s$.

Also $\exists\ \xi \in \mathcal{S}(F^\times)$ s.t. $Z(s,\xi,\chi) \equiv 1$. ( Take $\xi = \begin{cases} \chi^{-1} & \text{on } \mathcal{O}_F^\times \\ 0 & \text{o.w.} \end{cases}$ )

$\Rightarrow$ g.c.d $= 1$. i.e. $L(s,\pi\times\chi) = 1$.

(B) $\pi$ a generic member of the principal series.

more precisely $\pi = \pi_{\mu_1,\mu_2}$ s.t. $\mu_1\mu_2^{-1} \neq \mathbb{1}, |\ |^{\pm 1}$.

Recall space of $\mathcal{K}(\pi) = \left\{ |x|^{\frac{1}{2}} \left(\mu_1(x)\varphi_1(x) + \mu_2(x)\varphi_2(x)\right) \;\middle|\; \varphi_1,\varphi_2 \in \mathcal{S}(F) \right\}$

$$Z(s,\xi,\chi) = \int_{F^\times} \chi(x)|x|^s \left(\mu_1(x)\varphi_1(x) + \mu_2(x)\varphi_2(x)\right) d^\times x$$

$$= Z^{Tate}(s,\chi\mu_1,\varphi_1) + Z^{Tate}(s,\chi\mu_2,\varphi_2).$$

no common factor

$\overset{\mu_1\neq\mu_2}{\Longrightarrow}$ g.c.d $= L(s,\pi\times\chi) = L(s,\chi\mu_1)\,L(s,\chi\mu_2)$

(C) $\pi = \pi_{\mu_1,\mu_2}$ special repn. $\mu_1\mu_2^{-1} = |x|$. (note $\pi_{\mu_1,\mu_2} \simeq \pi_{\mu_2,\mu_1}$)

$$\mathcal{K}(\pi) = \left\{ |x|^{\frac{1}{2}} \mu_1(x)\varphi_1(x) \;\middle|\; \varphi_1 \in \mathcal{S}(F) \right\}$$

Similar to (B)

$\Rightarrow \quad L(s,\pi\times\chi) = L(s,\chi\mu_1).$

(D) $\pi = \pi_{\mu_1, \mu_2}$ principal series $\mu_1 = \mu_2$ valuation (additive)

$$K(\pi) = \left\{ |x|^{\frac{1}{2}} \mu_1(x) \left( \varphi_1(x) + \varphi_2(x)\, v(x) \right) \;\middle|\; \varphi_1, \varphi_2 \in \mathcal{S}(F) \right\}$$

$$Z(s, \xi, \chi) = Z^{Tate}(s, \chi\mu_1, \varphi_1) + \boxed{\int_{F^\times} \chi\mu_1(x) |x|^s \varphi_2(x)\, v(x)\, d^\times x}.$$

Set $\lambda := \chi\mu_1$.

Analytic property of $\boxed{\phantom{x}}$ is determined by (Assume $\varphi_2(0) \neq 0$. Otherwise entire)

$$\int_{\substack{x \in F^\times \\ |x| \le 1}} \lambda(x)\, v(x)\, |x|^s\, d^\times x \qquad (**)$$

$$= \sum_{n=0}^{\infty} \int_{\varpi^n \mathcal{O}_F^\times} \lambda(x)\, n\, q^{-ns}\, d^\times x = \sum_{n=0}^{\infty} \boxed{n\, q^{-ns}\, \lambda(\varpi)^n} \int_{\mathcal{O}_F^\times} \lambda(x)\, d^\times x$$

If $\lambda$ ramified, $(**) = 0$.

If $\lambda$ unramified, $(**) = \sum_{n=0}^{\infty} n\, q^{-ns}\, \lambda(\varpi)^n = \dfrac{q^{-s}\lambda(\varpi)}{(1 - q^{-s}\lambda(\varpi))^2}$

$$= q^{-s}\lambda(\varpi) \cdot L(s, \lambda)^2$$

Also can show $\exists\ \xi$ s.t. $Z(s,\xi,\chi) = L(s,\chi\mu_1)^2$.

$\Rightarrow$ g.c.d. $L(s,\pi\times\chi) = L(s,\chi\mu_1)^2$.

Thus can unify the principal series.

$$L(s,\pi\times\chi) = L(s,\chi\mu_1)\,L(s,\chi\mu_2).$$

<u>Thm 10</u> $\pi$ irred adm. (not character) repn. of $GL_2(F)$

$\chi$ ~ $GL_1(F)$

① $\dfrac{Z(s,W,\chi)}{L(s,\pi\times\chi)} \in \mathbb{C}[q^{-s},q^{s}]$. $\forall\ W \in \mathcal{W}(\pi)$.

② $\exists\ W \in \mathcal{W}(\pi)$ s.t. $\dfrac{Z(s,W,\chi)}{L(s,\pi\times\chi)} \equiv 1$.

Functional equation :

$$Z(1-s,\ \pi(w)W,\ \omega_\pi \chi^{-1}) = \gamma(s, \pi\times\chi, \tau)\, Z(s, W, \chi)$$

$\Rightarrow$ Defn : $\varepsilon$ - factor

$$\underbrace{\boxed{\frac{Z(1-s,\ \pi(w)W,\ \omega_\pi \chi^{-1})}{L(1-s,\ \pi\times \omega_\pi \chi^{-1})}}}_{\text{entire}} = \varepsilon(s, \pi\times\chi, \tau) \underbrace{\boxed{\frac{Z(s, W, \chi)}{L(s, \pi\times\chi)}}}_{\text{entire}}$$

$\varepsilon(s, \chi, \tau)$ is monomial in $q^{-s}$.

( Take $W$ s.t. RH box is 1. $\Rightarrow$ $\varepsilon(s, \chi, \tau) \in \mathbb{C}[q^s, q^{-s}]$.

Take $W$ s.t. LH box is 1 $\Rightarrow$ $\varepsilon(s, \chi, \tau) \in \mathbb{C}[q^s, q^{-s}]^\times$

so it is a unit. )